Closing Tue: TN 1, 2, 3
Closing Next Thu: TN 4, 5
Final: Sat, June 3rd, 5:00-7:50pm, KANE 130
Eight pages of questions; cumulative.

TN 1: Tangent Line Error Bounds

Def'n: The 1st Taylor polynomial for f(x) based at b is

$$T_1(x) = f(b) + f'(b)(x-b)$$

Entry Task: Find the 1st Taylor polynomial for $f(x) = \ln(x)$ at b = 1. Then use a calculator to fill in the table.

	$f(x)$	$T_1(x)$	$\lvert f(x) - T_1(x)\rvert$
$x = 1$			
$x = 1.2$			
$x = 1.4$			
$x = 0.9$			

Warm up: An <u>upper **bound**</u>, M, is any number that is always bigger than a function on a given interval.

Examples: Find any upper **bound** on the given intervals

1. $|\sin(5x)|$ on $[0,2\pi]$

2. $|x-3|$ on $[1,5]$

3. $\left|\frac{1}{(2-x)^3}\right|$ on $[-1,1]$

4. $|\sin(x)+\cos(x)|$ on $[0,2\pi]$

5. $\left|\cos(2x) + e^{-2x} + \frac{6}{x}\right|$ on $[1,4]$

Tangent Linear Error Bound Thm

(Taylor's inequality)

On a given interval [a,b],

if $|f''(x)| \le M$, then

$$|f(x) - T_1(x)| \le \frac{M}{2}|x-b|^2$$

(on the interval).

Note:

M = any upper bound on f''(x).

$|x-b| =$ "dist x is away from b".

Proof for x > b:

Start with $f(x) - f(b) = \int_b^x f'(t)dt$.

Using by parts (u = f'(t), dv = dt):

$$f(x) - f(b) = f'(b)(x-b) - \int_b^x (t-x)f''(t)dt$$

$$f(x) - f(b) - f'(b)(x-b) = \int_b^x (x-t)f''(t)dt$$

Thus,

$$|f(x) - T_1(x)| = \left|\int_b^x (x-t)f''(t)dt\right|$$

Then note

$$\left|\int_b^x (x-t)f''(t)dt\right| \le \int_b^x (x-t)|f''(t)|dt$$

$$\le M\int_b^x (x-t)dt$$

$$= \frac{M}{2}(x-b)^2$$

To use the error bound theorem:

1. Find f''(t).
2. Find ***any*** upper bound for |f''(t)|. Call this M.
3. Use the theorem.
4. And plug in x = "an endpoint" to get a single number for a worst case upper bound.

Two types of error bound questions in the current homework:

A) Given interval, find error bound.
B) Given error bound, find interval.

Example:

Let f(x) = ln(x). Find the 1^{st} Taylor polynomial based at b=1.

1. Find a bound on the error over the interval J = [1/2, 3/2]
2. Find an interval around b = 1 where the error is less than 0.01.

Example (you do):

Let $f(x) = x^{1/3}$ and $b = 8$. Find the 1^{st} Taylor Polynomial.

Use Taylor's inequality to give a bound on the error over the interval $J = [7,9]$.

(TN 2/ 3): Higher Order Approx.

The **2^{nd} Taylor Polynomial** (or quadratic approximation) is given by

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$$

The **quadratic error bound theorem** (Taylor's inequality) states:

on a given interval [a,b],

if $|f'''(x)| \leq M$, then

$$|f(x) - T_2(x)| \leq \frac{M}{6}|x-b|^3$$

Example:

Find the 2^{nd} Taylor polynomial for $f(x) = x^{1/3}$ based at b = 8 and find the error bound on the interval J = [7,9].

Taylor Approximation Idea:
If two functions have all the same derivative values, then they are the same function (up to a constant).

To explain, let's compare derivatives of f(x) and $T_2(x)$ at b.

$$T_2(x) = f(b) + f'(b)(x-b) + \tfrac{1}{2} f''(b)(x-b)^2$$

$$T_2'(x) = f'(b) + \frac{1}{2} f''(b)2(x-b)$$

$$= f'(b) + f''(b)(x-b)$$

$$T_2''(x) = f''(b)$$

$$T_2'''(x) = 0$$

Now plug in x = b to each of these.
What do you see?

Questions:
Why did we need a ½ ?

What would $T_3(x)$ look like?

n^{th} Taylor polynomial

$$f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{3!}f'''(b)(x-b)^3 + \cdots + \frac{1}{n!}f^{(n)}(b)(x-b)^n$$

In sigma notation:

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

Taylor's Inequality (error bound):
on a given interval [a,b],
if $|f^{(n+1)}(x)| \leq M$, then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1}$$

Side Note:

For a fixed constant, *a*, the expression $\frac{a^k}{k!}$ goes to zero as k goes to infinity.

So the expression $\frac{1}{(n+1)!}|x-b|^{n+1}$, will always go to zero as *n* gets bigger.

Which means that the error goes to zero, unless something unusual is happening with M, which will see in examples later.
